Problem 1

I. Let the vacuum permittivity be ε_0. Answer the following questions.

(1) As shown in Fig. 1, a point charge Q is placed at the origin O in a vacuum space. Find the electric field and the electric potential at a position $(x,\ y,\ z)$. Let the electric potential at infinity be 0.

(2) The point charge in Fig. 1 is replaced with an infinitely-long line charge along the z direction (line charge density λ) (see Fig. 2). Find the electric field and the electric potential at a position $(x,\ y,\ z)$. Let the electric potential at the point $(a,\ 0,\ 0)$ $(a > 0)$ be 0.

Next, as shown in Fig. 3, several line charges are placed in a vacuum space. The line charge density of the line charges on the lines $x = \pm a,\ \ y = 0$ is λ, whereas the line charge density of the line charges on the lines $x = 0,\ \ y = \pm a$ is $-\lambda$. Assume $\lambda > 0$. Letting the electric potential at the origin O be 0, the electric potential around the z axis, $\phi(x,\ y,\ z)$, can be approximated by Eq. (i).

$$\phi(x, y, z) = \frac{\lambda}{\pi\varepsilon_0}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2\right] \quad \text{(i)}$$

In addition, a uniform magnetic field with a magnetic flux density of B is applied in the positive direction along the x axis. Let us consider a charged particle (mass m, charge $q > 0$) at a position $(x,\ y,\ z)$ around the origin O. Neglect any influence of the particle motion on the magnetic field and the line charges.

(3) Find the force $\boldsymbol{F} = (F_x,\ F_y,\ F_z)$ acting on the charged particle, and write down the equations of motion for the particle.

(4) At $t = 0$, the particle is at the origin O, and its velocity and acceleration are $\boldsymbol{v}_0 = (0,\ v_y^0,\ 0)$ $(v_y^0 \neq 0)$ and $\boldsymbol{a}_0 = (0,\ 0,\ 0)$, respectively. The mass m should satisfy the condition $m < m_0$ so that the charged particle moves while being bound around the z axis for $t > 0$. Find m_0.

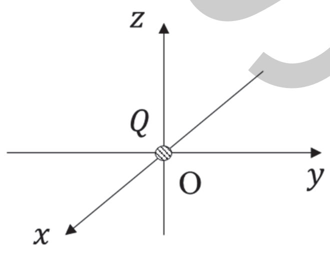

Fig. 1

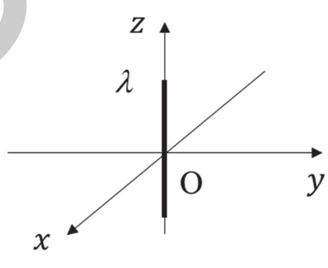

Fig. 2

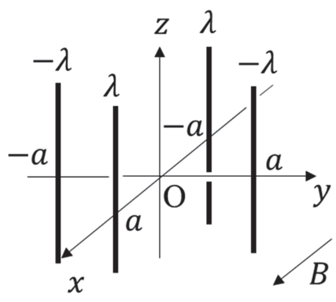

Fig. 3

II. Let the vacuum permittivity and the vacuum permeability be ε_0 and μ_0, respectively. Answer the following questions.

As shown in Fig. 4, a cylindrical conductor of radius a, which is made of a uniform material with an electrical conductivity of σ, is placed in a vacuum space. The central axis of the conductor is set as the z axis, and a current I is flowing uniformly through the conductor in the positive z direction. Suppose that the conductor is sufficiently longer than the radius a.

(1) Find the direction and the magnitude of the magnetic flux density at a position with a distance r from the central axis of the conductor. The conductor is assumed to be non-magnetic with a permeability of μ_0.

(2) Find the direction and the magnitude of the electric field inside the conductor. Also, find the Joule heating generated per unit time per unit length in the conductor.

(3) Find the direction and the magnitude of the Poynting vector at the surface of the conductor. Also, find the electromagnetic energy coming in and out from the conductor through its surface per unit time per unit length. The Poynting vector at a position $\boldsymbol{r}$, $\boldsymbol{S}(\boldsymbol{r})$, is given by $\boldsymbol{S}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r})$, where $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$ are the electric field and the magnetic field at the position $\boldsymbol{r}$, respectively.

Consider a parallel-plate capacitor with circular electrodes of radius b, spaced apart with a distance d, placed in a vacuum space. An electric charge Q_0 is stored in the capacitor. The capacitor is connected to a resistor R through a switch (see Fig. 5). The capacitor starts to discharge when the switch is closed, at $t = 0$. $Q(t)$ is the electric charge stored in the capacitor at time t. Suppose that b is sufficiently larger than d so that the edge effect is negligible.

(4) Find the capacitance C of the capacitor and the energy U_1 stored in the capacitor before discharge starts.

(5) Express the electric filed between the electrodes and the magnetic flux density at the distance of b from the central axis of the capacitor at time t, using $Q(t)$.

(6) Using the Poynting vector, find the net electromagnetic energy U_2 coming in and out from the capacitor in a sufficiently long time period after the discharge starts.

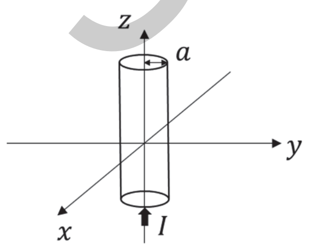

Fig. 4

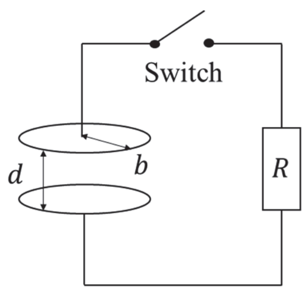

Fig. 5