**Problem 3**

I. Answer the following questions on information theory. Let $X = \{0, 1\}$ be a memoryless source of information, whose $i$-th signal is denoted as $X_i$ $(i = 1, 2, 3, \cdots)$. The probability of being $X_i = 0$ is $p$ and that of being $X_i = 1$ is $1 - p$. You may use the following approximations: $\log_2 3 \approx 1.6$, $\log_2 5 \approx 2.3$, and $\log_2 7 \approx 2.8$.

1. Obtain the entropy $H(X)$ assuming $p = 0.75$.

2. Assuming $p = 0.75$, let us efficiently encode four values $\{00, 01, 10, 11\}$, which are the combinations of two successive signals of $X$. Show an example of code words, and calculate its average symbol length.

Consider a memoryless communication channel $C$, whose input is $X$. The output of $C$ is $Y = \{0, 1\}$, whose $i$-th signal is denoted as $Y_i$. There are an 80% chance of $Y_i = X_i$ and a 20% chance of $Y_i = 1$ irrespective of $X_i$.

3. Obtain the entropy $H(Y)$ and the mutual information $I(X; Y)$ assuming $p = 0.75$.

4. Answer whether the value of $p$ that maximizes $I(X; Y)$ is larger or smaller than 0.5, and briefly explain the reason for it.
II. Answer the following questions on signal processing. Let time $t$ and angular frequency $\omega$ be real, and $j$ be the imaginary unit. Denote the complex conjugate of a complex number $a$ as $a^*$. The Fourier transform $X(\omega)$ of a complex function $x(t)$ and its inverse Fourier transform are defined as follows:

$$ X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt $$

$$ x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega $$

(i) Show that $\mathcal{F}^{-1}[X^*(\omega)] = x^*(-t)$ holds.

(ii) Show that $X^*(\omega) = X(-\omega)$ holds if $x(t)$ is a real function.

Let us denote the impulse response of an analog filter $A$ as a real-valued function $f(t)$. Since the response of $A$ satisfies causality, $f(t) = 0$ for $t < 0$. Denoting the real part and imaginary part of $F(\omega) = \mathcal{F}[f(t)]$ as $F_1(\omega)$ and $F_2(\omega)$, respectively, $F(\omega) = F_1(\omega) + jF_2(\omega)$.

(3) Express $f_1(t) = \mathcal{F}^{-1}[F_1(\omega)]$ using $f(t)$.

(4) Express $f_2(t) = \mathcal{F}^{-1}[F_2(\omega)]$ using $f(t)$.

(5) If $F_1(\omega)$ is known and $F_2(\omega)$ is unknown, $F_2(\omega)$ can be derived from $F_1(\omega)$ by using Fourier transform and inverse Fourier transform. Describe the procedures for the derivation in about three lines. You may use figures and equations if necessary.

Consider a real-valued signal $s_1(t)$, whose angular frequency band is $|\omega| \leq \omega_B$, i.e., $\mathcal{F}[s_1(t)] = S_1(\omega) = 0$ for $|\omega| > \omega_B$. Let us modulate a carrier wave at an angular frequency of $\omega_c$ $(\gg \omega_B)$ by this signal.

(6) Express the Fourier transform of a real-valued signal $d(t) = s_1(t) \cos \omega_c t$, i.e., $D(\omega) = \mathcal{F}[d(t)]$ using $S_1(\omega)$. Also, show that the angular frequency band of $D(\omega)$ is $\omega_c - \omega_B \leq |\omega| \leq \omega_c + \omega_B$.

(7) If $s_1(t)$ is known, we can prepare an appropriate real-valued signal $s_2(t)$ and generate a real-valued signal $u(t) = s_1(t) \cos \omega_c t + s_2(t) \sin \omega_c t$ such that the angular frequency band of $U(\omega) = \mathcal{F}[u(t)]$ is limited to $\omega_c \leq |\omega| \leq \omega_c + \omega_B$. Describe the procedures for the derivation of $s_2(t)$ from $s_1(t)$ in about three lines. You may use figures and equations if necessary.