Problem 1

I.

Consider the following steps (1)−(3) for obtaining the charge density induced on a conducting sphere located in a uniform electric field. Let the vacuum permittivity be ε_0. Answer the following questions.

(1) Two point charges are located at point A$(L, 0, 0)$ and point B$(-L, 0, 0)$ ($L > 0$) in vacuum space described by a Cartesian coordinate system (x, y, z) as shown in Fig. 1. They have charges with equal magnitude but with opposite sign, creating an electric field whose x-direction component at the origin is E_0 (> 0).

(1-i) Find the amount of charge located at A and B, respectively.

(1-ii) Find the electric potential at point C$(a, \sqrt{R^2 - a^2}, 0)$ generated by the point charge located at A. Here, $|a| \leq R < L$ as shown in Fig. 1 and the electric potential at infinity is set to be zero.

(2) Next, as shown in Fig. 2, a grounded conducting sphere with radius R $(R < L)$ centered at the origin is placed together with the two point charges at A and B.

(2-i) The point charges located at A and B induce the surface charges on the conducting sphere. The electric field and the electric potential outside the conducting sphere can be obtained by replacing the surface charges induced on the conductor with virtual "image charges" inside the sphere that satisfy the boundary condition of zero electric potential on the surface of the conductor sphere. Find the location and the amount of the "image charge" corresponding to the point charge located at A and B, respectively.

(2-ii) Show that the total charge induced on the conducting sphere is zero.

(3) Then, assume $L \gg R$ so that the electric field given in Question (1) can be approximated by a uniform electric field $\boldsymbol{E_0} = (E_0, 0, 0)$ around the origin.

(3-i) Show that the "image charges" obtained in Question (2-i) can be expressed as a single electric dipole located at the origin with the dipole moment $\boldsymbol{P} = (4\pi\varepsilon_0 R^3 E_0, 0, 0)$.

(3-ii) Suppose that the conducting sphere is removed and replaced by the "image dipole" given in Question (3-i). Employ a polar coordinate system (r, θ, ϕ) as shown in Fig. 3 and find the tangential component $E_{0\theta}$ and the normal component E_{0r} of the uniform electric field $\boldsymbol{E_0}$ at point D(R, θ, ϕ), which corresponds to the surface of the removed conducting sphere.

(3-iii) An electric dipole located at the origin creates an electric potential $V(\boldsymbol{r})$ at location $\boldsymbol{r}$ as

$$V(\boldsymbol{r}) = \frac{\boldsymbol{P} \cdot \boldsymbol{r}}{4\pi\varepsilon_0 |\boldsymbol{r}|^3}.$$

Show that the total electric field created by the electric dipole given in Question (3-i) and the uniform electric field $\boldsymbol{E_0}$ satisfies the boundary condition of electric field on the surface of the conducting sphere. You may use the following formula for vector analysis.

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi$$

(3-iv) Find the surface charge density induced on the surface of the conducting sphere as a function of angle θ.

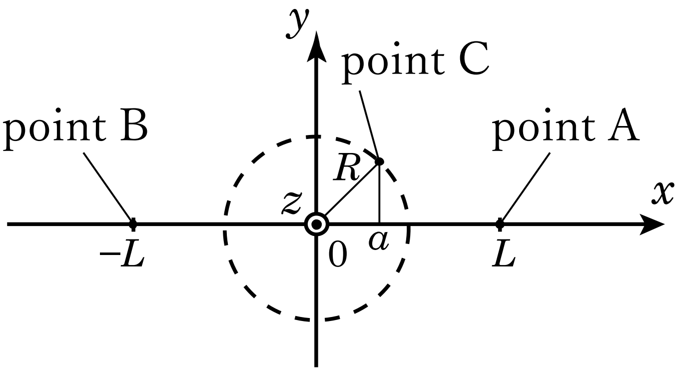

Fig. 1

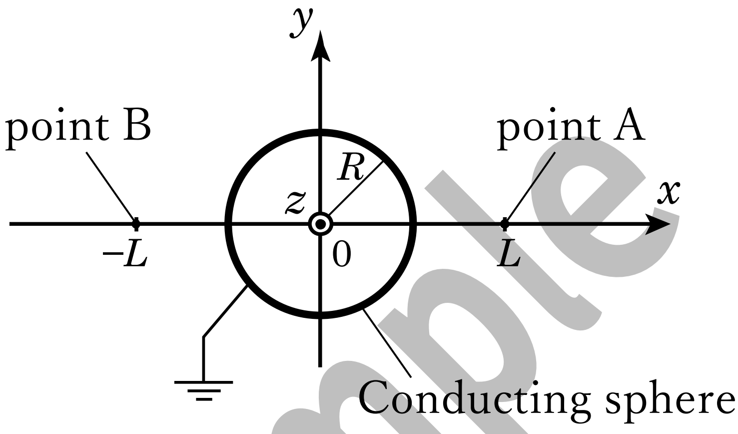

Fig. 2

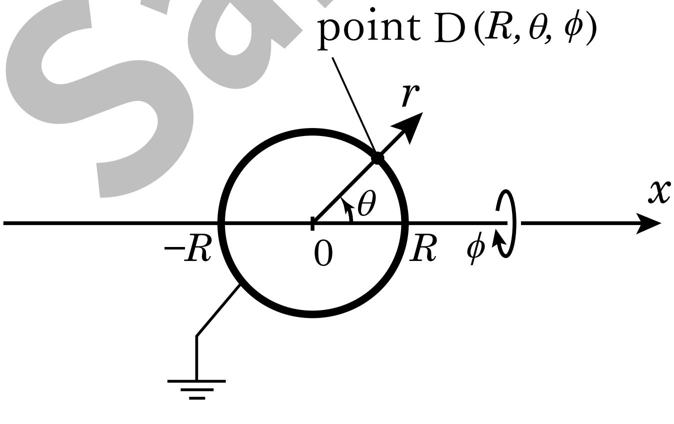

Fig. 3

II.

Consider that an infinitely long straight wire conductor with a negligible thickness is located on the z-axis in vacuum space described by a Cartesian coordinate system (x, y, z) as shown in Fig. 4. A current I flows in the straight wire conductor in the positive z-direction. A square-shaped one-turn coil with a side length of $2a$ is located with its center at $(\sqrt{2}a, 0, 0)$ and the side AB and side CD are aligned in parallel to the z-axis. As shown in Fig. 5, the coil can be tilted so that the angle between the x-axis and the unit normal vector $\boldsymbol{n}$ of the coil plane is set to be θ. Note that the coil shown in Fig. 4 has the angle $\theta = \pi/2$. The coil consists of a wire with a negligible thickness and no current flows in the coil. Let the vacuum permeability be μ_0. Answer the following questions.

(1) Find the x-, y-, and z-components of the magnetic flux density at an arbitrary point $\mathrm{P}(x, y, z)$.

(2) Find the magnetic flux passing through the coil when the angle θ is 0 and $\pi/2$, respectively. Here, the magnetic flux density in the direction of the normal vector $\boldsymbol{n}$ is defined to be positive.

(3) Show that the vector potential $\boldsymbol{A}$ generated by the current I is given as $\boldsymbol{A} = \boldsymbol{A}_0 + \nabla\varphi$, where

$$\boldsymbol{A}_0 = \left(0, 0, -\frac{\mu_0 I}{2\pi}\log\sqrt{x^2 + y^2}\right)$$

and φ is an arbitrary scalar function of x, y and z.

(4) Express the mutual inductance between the straight wire conductor and the coil as a function of θ. In derivation, you may use the vector potential considered in Question (3).

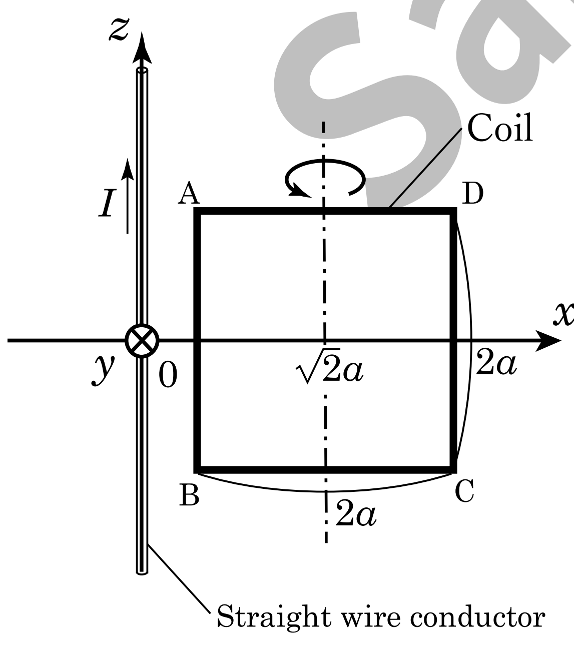

Fig. 4

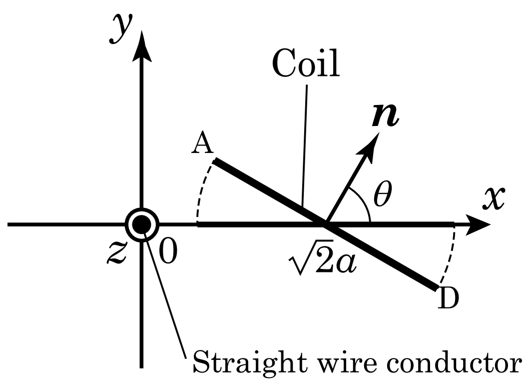

Fig. 5