Problem 3

I.

Answer the following questions on information theory. Let S be a Markov information source that has three states $\{s_0, s_1, s_2\}$. For each trial, S makes a transition to the next state according to the transition probability matrix T given by the following equation.

$$T = \begin{bmatrix} p(s_0|s_0) & p(s_1|s_0) & p(s_2|s_0) \\ p(s_0|s_1) & p(s_1|s_1) & p(s_2|s_1) \\ p(s_0|s_2) & p(s_1|s_2) & p(s_2|s_2) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0.8 & 0 & 0.2 \end{bmatrix}$$

During each transition, S outputs the next state. You may use the following approximations when necessary: $\log_2 3 = 1.58$, $\log_2 5 = 2.32$, and $\log_2 7 = 2.81$.

(1) Draw the state transition diagram of S.

(2) Obtain the probabilities P_0, P_1, and P_2 of being in the states s_0, s_1, and s_2, respectively, after a sufficient number of trials.

(3) Obtain the overall entropy of S. Express the answer using P_0, P_1, and P_2 defined in Question (2).

(4) Consider encoding two successive outputs of S into code words.

(4-i) Design a set of code words that maximizes the encoding efficiency.

(4-ii) Describe how to quantify the encoding efficiency of the set of code words designed in Question (4-i).

Problem 3

II.

Answer the following questions on signal processing. The Fourier transform $F(\omega)$ of a continuous-time signal $f(t)$ is given by

$$\boldsymbol{F}(\boldsymbol{\omega}) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt \ , \tag{i}$$

where t $(-\infty < t < \infty)$ is the time, ω is the angular frequency, and j is the imaginary unit. Its inverse Fourier transform is given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega \ . \tag{ii}$$

The unit impulse function $\delta(\boldsymbol{t})$ is expressed as

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1 \ . \tag{iii}$$

The complex Fourier series of a periodic function $g(t)$ with a period T is expressed as

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} \ , \ \text{where } c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) e^{-j\frac{2\pi}{T}nt} dt \quad (n\text{: integer}) \ . \tag{iv}$$

(1) Let $X(\omega)$ be the Fourier transform of a real continuous-time signal $x(t)$. Show that its complex conjugate $X^{\star}(\omega)$ is equal to $X(-\omega)$.

(2) Obtain the Fourier transform of the unit impulse function.

(3) Assume that a signal $x(t)$ is input to a low-pass filter and the output signal $y(t)$ is obtained, and that the low-pass filter has an ideal frequency response $H(\omega)$ defined by

$$H(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases} \ , \tag{v}$$

where ω_c (> 0) is the angular cutoff frequency. Obtain the output signal $y(t)$ assuming that $x(t)$ is the unit impulse. Also sketch an outline of $y(t)$, and indicate the value of $y(0)$ and value(s) of t where $y(t) = 0$.

(4) A periodic impulse train $d(t)$ with a period of T (> 0) is expressed as

$$d(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \, . \tag{vi}$$

(4-i) Express $d(t)$ in the form of the complex Fourier series.

(4-ii) Show that $D(\omega)$ consists of a periodic impulse train, where $D(\omega)$ is the Fourier transform of $d(t)$.

(5) Consider the sampling of a real continuous-time signal $x(t)$.

(5-i) Explain how to recover $x(t)$ from the sampled signals in a few lines. You may use Eqs. (iv) to (vi) if necessary.

(5-ii) Describe the condition to recover $x(t)$ from the sampled signals.