Problem 3

I. Consider a binary communication channel Γ shown in Fig. 1. Here, Γ is a steady-state memoryless channel, where the input is $X = \{0, 1\}$ and the output is $Y = \{0, 1\}$. Suppose that the bit error probability for an input symbol of 0 is p, and the bit error probability for an input symbol of 1 is q. Probabilities p and q satisfy $0 \leq p \leq 1$ and $0 \leq q \leq 1$, respectively. Answer the following questions. In this problem, use $f(0) = 0$ for $f(x) = x \log_2 x$, if necessary. Provide the derivation process for your answers.

(1) Assume that X is a memoryless source, where the probabilities of an input symbol of 0 and 1 are both $1/2$. When $p = 1/4$ and $q = 1/8$, obtain the following values. Use $\log_2 3 = 1.6$, $\log_2 5 = 2.3$, and $\log_2 7 = 2.8$, if necessary.

(1-i) $H(X)$: entropy of the input X.

(1-ii) $H(Y)$: entropy of the output Y.

(1-iii) $I(X;Y)$: mutual information between X and Y.

(2) Consider the channel capacity C of Γ when $p = q$.

(2-i) Obtain C as a function of p.

(2-ii) Obtain p that maximizes C.

(3) Consider the case when the channel capacity C of Γ becomes 0. Obtain the equation that p and q satisfy.

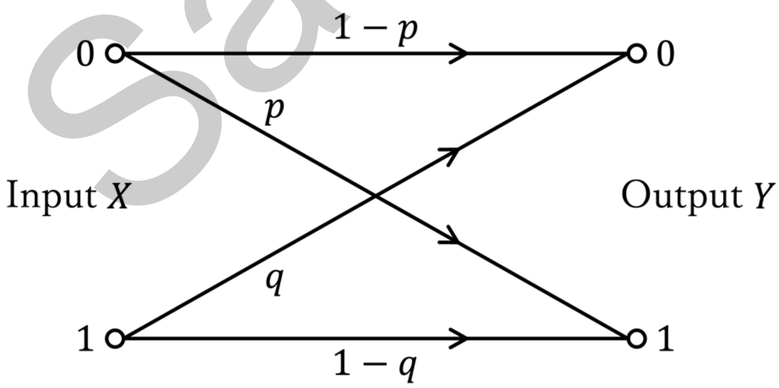

Fig. 1

II. The Fourier transform $F(\omega)$ of a continuous-time signal $f(t)$ is given by

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt,$$

where time t and angular frequency ω are real numbers, and j is the imaginary unit. Its inverse Fourier transform is given by

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega.$$

Answer the following questions. If necessary, use Parseval's theorem for the Fourier transform, which is given by

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega.$$

Provide the derivation process for your answers.

(1) Briefly describe the requirements for a signal to which the Fourier transform can be applied.

(2) Consider the Fourier transform of $F(t)$, which is the continuous-time signal obtained by substituting $\omega = t$ to $F(\omega)$. Then $\mathcal{F}[F(t)]$, which is the Fourier transform of $F(t)$, satisfies $\mathcal{F}[F(t)] = \alpha f(-\omega)$, where α is a constant. Obtain the constant α.

(3) Obtain the Fourier transform $X_1(\omega)$ of the continuous-time signal $x_1(t)$, which is given by

$$x_1(t) = \begin{cases} 1 & (-1 \le t \le 1) \\ 0 & (t < -1, 1 < t). \end{cases}$$

(4) Suppose that $y(t)$ is the output signal when a continuous-time signal $x_2(t)$ is input to a linear system A. Here, $x_2(t)$ and the impulse response $h(t)$ of A are respectively given by

$$x_2(t) = \begin{cases} \dfrac{\sin(t-1)}{t-1} & (t \ne 1) \\ 1 & (t = 1), \end{cases}$$

$$h(t) = \begin{cases} e^{-t} & (t \ge 0) \\ 0 & (t < 0). \end{cases}$$

(4-i) Obtain the Fourier transform $X_2(\omega)$ of $x_2(t)$.

(4-ii) Obtain the energy E_y of $y(t)$, which is defined as

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt.$$