Problem 3

I. Answer the following questions on information theory. Let $X = \{0, 1\}$ be a memoryless source of information, whose i-th signal is denoted as X_i $(i = 1, 2, 3, \cdots)$. The probability of being $X_i = 0$ is p and that of being $X_i = 1$ is $1 - p$. You may use the following approximations: $\log_2 3 = 1.6$, $\log_2 5 = 2.3$, and $\log_2 7 = 2.8$.

(1) Obtain the entropy $H(X)$ assuming $p = 0.75$.

(2) Assuming $p = 0.75$, let us efficiently encode four values $\{00, 01, 10, 11\}$, which are the combinations of two successive signals of X. Show an example of code words, and calculate its average symbol length.

Consider a memoryless communication channel C, whose input is X. The output of C is $Y = \{0, 1\}$, whose i-th signal is denoted as Y_i. There are an 80% chance of $Y_i = X_i$ and a 20% chance of $Y_i = 1$ irrespective of X_i.

(3) Obtain the entropy $H(Y)$ and the mutual information $I(X; Y)$ assuming $p = 0.75$.

(4) Answer whether the value of p that maximizes $I(X; Y)$ is larger or smaller than 0.5, and briefly explain the reason for it.

II. Answer the following questions on signal processing. Let time t and angular frequency ω be real, and j be the imaginary unit. Denote the complex conjugate of a complex number a as a^*. The Fourier transform $X(\omega)$ of a complex function $x(t)$ and its inverse Fourier transform are defined as follows:

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{i}$$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{ii}$$

(1) Show that $\mathcal{F}^{-1}[X^*(\omega)] = x^*(-t)$ holds.

(2) Show that $X^*(\omega) = X(-\omega)$ holds if $x(t)$ is a real function.

Let us denote the impulse response of an analog filter A as a real-valued function $f(t)$. Since the response of A satisfies causality, $f(t) = 0$ for $t < 0$. Denoting the real part and imaginary part of $F(\omega) = \mathcal{F}[f(t)]$ as $F_1(\omega)$ and $F_2(\omega)$, respectively, $F(\omega) = F_1(\omega) + jF_2(\omega)$.

(3) Express $f_1(t) = \mathcal{F}^{-1}[F_1(\omega)]$ using $f(t)$.

(4) Express $f_2(t) = \mathcal{F}^{-1}[F_2(\omega)]$ using $f(t)$.

(5) If $F_1(\omega)$ is known and $F_2(\omega)$ is unknown, $F_2(\omega)$ can be derived from $F_1(\omega)$ by using Fourier transform and inverse Fourier transform. Describe the procedures for the derivation in about three lines. You may use figures and equations if necessary.

Consider a real-valued signal $s_1(t)$, whose angular frequency band is $|\omega| \le \omega_B$, i.e., $\mathcal{F}[s_1(t)] = S_1(\omega) = 0$ for $|\omega| > \omega_B$. Let us modulate a carrier wave at an angular frequency of ω_c $(\gg \omega_B)$ by this signal.

(6) Express the Fourier transform of a real-valued signal $d(t) = s_1(t) \cos \omega_c t$, i.e., $D(\omega) = \mathcal{F}[d(t)]$ using $S_1(\omega)$. Also, show that the angular frequency band of $D(\omega)$ is $\omega_c - \omega_B \le |\omega| \le \omega_c + \omega_B$.

(7) If $s_1(t)$ is known, we can prepare an appropriate real-valued signal $s_2(t)$ and generate a real-valued signal $u(t) = s_1(t) \cos \omega_c t + s_2(t) \sin \omega_c t$ such that the angular frequency band of $U(\omega) = \mathcal{F}[u(t)]$ is limited to $\omega_c \le |\omega| \le \omega_c + \omega_B$. Describe the procedures for the derivation of $s_2(t)$ from $s_1(t)$ in about three lines. You may use figures and equations if necessary.