Problem 3

I.

Answer the following questions on information theory. Suppose that we transmit information by using a time-discrete communication channel C, whose input and output are designated as $X \in \{-1,1\}$ and $Y \in \{-1,1\}$, respectively. The input and output relation of the i-th communication via C is represented as $Y_i = Z_i \times X_i$ ($i = 1, 2, \cdots$), where $\times$ means the multiplication of integers. $Z_i \in \{-1,1\}$ is an internal state of the channel at the i-th communication, and its value can change depending on the current or past states of the input and on the past states of the output. Both sender and receiver are unable to observe the value of Z_i directly although they can have knowledge about how Z_i changes depending on the input and output. Use the logarithm base 2 for your answers of the following questions. You may also use the following approximations upon necessity: $\log_2 3 = 1.585$, $\log_2 5 = 2.322$, and $\log_2 7 = 2.807$.

(1) Let X_i be an ideally independent random variable that takes $X_i = 1$ with probability μ and $X_i = -1$ with probability $1 - \mu$. Assume that Z_i becomes 1 with probability 1 when $X_i = 1$ and that it takes either 1 or -1 with equal probability when $X_i = -1$.

   (1-i) Obtain the entropies $H[X]$ and $H[Y]$ and the conditional entropy $H[Y|X]$ of C.

   (1-ii) Obtain the channel capacity of C.

(2) Assume that Z_1 takes either 1 or -1 with equal probability and that, for $i \geq 2$, the value of Z_i becomes the same as the previous output value Y_{i-1} with probability 1 as $Z_i = Y_{i-1}$. Obtain the maximum bits that can be transmitted by using this channel n times.

(3) Assume that Z_i takes either 1 or -1 with equal probability when i is odd and that Z_i keeps its previous value with probability 1 as $Z_i = Z_{i-1}$ when i is even. Obtain the channel capacity of C and show a code that can achieve the capacity.

(4) Assume that $Z_1 = 1$ with probability 1 and that, for $i \geq 2$, the value of Z_i becomes the same as the previous input value X_{i-1} with probability 1 as $Z_i = X_{i-1}$. Let X_i be an ideally independent random variable that takes $X_i = 1$ with probability μ and $X_i = -1$ with probability $1 - \mu$. Calculate the probability q that $Y_i = 1$ at the stationary state for sufficiently large i.

II.

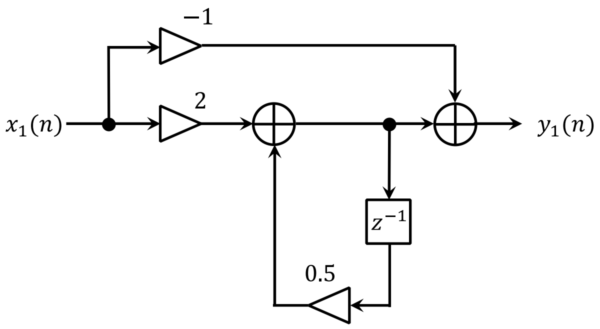
Answer the following questions on signal processing. Consider the two infinite impulse response systems shown in Figs. 1 and 2. $x_1(n)$ and $y_1(n)$ are the input and output signal sequences of system 1 in Fig. 1, respectively, and represent the signal values at time $nT(T > 0)$ for $n = 0,1,\cdots$. Similarly, $x_2(n)$ and $y_2(n)$ are the input and output sequences of system 2 in Fig. 2. The circuits consist of adders, coefficient multipliers, and delays, whose respective functions are described in Fig. 3.

(1) Obtain the impulse response of system 1, $h_1(n)$, and its *z*-transform $H_1(z)$.

(2) Calculate the frequency response of system 1 and explain the filtering function of this system on the input signal.

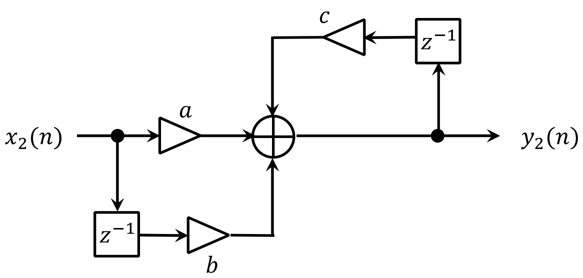
(3) Obtain the parameter values of a, b, and c that makes system 2 equivalent to system 1.

(4) Draw an equivalent circuit of system 2 that has a smaller number of delays than the original system 2 shown in Fig. 2.



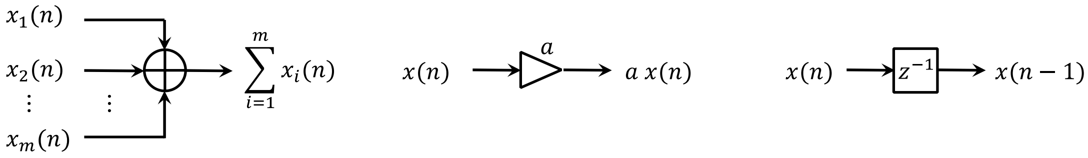
System 1

Fig. 1



System 2

Fig. 2



Adder

Coefficient multiplier

Delay

Fig. 3